

On Jacobian group and complexity of I -graph $I(n, k, l)$ through Chebyshev polynomials

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Abstract

We consider a family of I -graphs $I(n, k, l)$, which is a generalization of the class of generalized Petersen graphs. In the present paper, we provide a new method for counting Jacobian group of the I -graph $I(n, k, l)$. We show that the minimum number of generators of $Jac(I(n, k, l))$ is at least two and at most $2k + 2l - 1$. Also, we obtain a closed formula for the number of spanning trees of $I(n, k, l)$ in terms of Chebyshev polynomials. We investigate some arithmetical properties of this number and its asymptotic behaviour.

Key Words: Spanning tree, Jacobian group, I -graph, Petersen graph, Chebyshev polynomial

AMS Mathematics Subject Classification: 05C30, 39A10

1 Introduction

The notion of the Jacobian group of a graph, which is also known as the Picard group, the critical group, and the dollar or sandpile group, was independently introduced by many authors ([1], [2], [3], [4]). This notion arises as a discrete version of the Jacobian in the classical theory of Riemann surfaces. It also admits a natural interpretation in various areas of physics, coding theory, and financial mathematics. The Jacobian group is an important algebraic invariant of a finite graph. In particular, its order coincides with the number of spanning trees of the graph, which is known for some simplest graphs, such as the wheel, fan, prism, ladder, and Möbius ladder [5], grids [12], lattices [14], prism and anti-prism [15]. At the same time, the structure of the Jacobian is known only in particular cases [1], [3], [8], [17], [18], [19] and [20]. We mention that the number of spanning trees for circulant graphs is expressed in terms of the Chebyshev polynomials; it was found in [9], [10], and [11]. We show that similar results are also true for the I -graph $I(n, k, l)$.

The generalized Petersen graph $GP(n, k)$ has vertex set and edge set given by

$$\begin{aligned} V(P(n, k)) &= \{u_i, v_i \mid i = 1, 2, \dots, n\} \\ E(P(n, k)) &= \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} \mid i = 1, 2, \dots, n\}, \end{aligned}$$

where the subscripts are expressed as integers modulo n . The classical Petersen graph is $P(5, 2)$. The family of generalized Petersen graphs is a subset of so-called I -graphs. The I -graph $I(n, k, l)$ is a graph of the following structure

$$\begin{aligned} V(I(n, k, l)) &= \{u_i, v_i \mid i = 1, 2, \dots, n\} \\ E(P(n, k, l)) &= \{u_i u_{i+l}, u_i v_i, v_i v_{i+k} \mid i = 1, 2, \dots, n\}. \end{aligned}$$

where all subscripts are given modulo n .

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Since $I(n, k, l) = I(n, l, k)$ we will usually assume that $k \leq l$. In this paper we will deal with 3-valent graphs only. This means that in the case of even n and $l = n/2$ the graph under consideration has multiple edges. If $GCD(n, k, l) = m > 1$, then $I(n, k, l)$ is a union of m copies of the graph $I(n/m, k/m, l/m)$. If $m = 1$ and $GCD(k, l) = d$, then the graphs $I(n, k, l)$ and $I(n, k/d, l/d)$ are isomorphic [25], [22], [24]. So, in what follows, we assume k and l to be relatively prime. In the case of $l = 1$ it is easy to see that the graph $I(n, k, 1)$ coincides with the generalized Petersen graph $GP(n, k)$. The number of spanning trees and the structure of Jacobian group for the generalized Petersen graph were investigated in [27]. The spectrum of the I -graph was found in [28]. Even though the number of spanning trees of a given graph can be computed through eigenvalues of its Laplacian matrix, it is not easy to find the number of spanning trees for $I(n, k, l)$ using them. In this paper, we obtained a closed formula for the number of spanning trees for $I(n, k, l)$, investigate some its arithmetical properties of this number and provide its asymptotic behavior. Also, we suggest an effective way for calculating Jacobian of $I(n, k, l)$ and find sharp upper and lower bounds for the rank of $Jac(I(n, k, l))$.

2 Basic definitions and preliminary facts

Consider a connected finite graph G , allowed to have multiple edges but without loops. We endow each edge of G with the two possible directions. Since G has no loops, this operation is well defined. Let $O = O(G)$ be the set of directed edges of G . Given $e \in O(G)$, we denote its initial and terminal vertices by $s(e)$ and $t(e)$, respectively. Recall that a closed directed path in G is a sequence of directed edges $e_i \in O(G)$, $i = 1, \dots, n$ such that $t(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$ and $t(e_n) = s(e_1)$.

Following [2] and [4], the *Jacobian group*, or simply *Jacobian* $Jac(G)$ of a graph G is defined as the (maximal) abelian group generated by flows $\omega(e), e \in O(G)$, obeying the following two Kirchhoff laws:

K_1 : the flow through each vertex of G vanishes, that is $\sum_{e \in O, t(e)=x} \omega(e) = 0$ for all $x \in V(G)$;

K_2 : the flow along each closed directed path W in G vanishes, that is $\sum_{e \in W} \omega(e) = 0$.

Equivalent definitions of the group $Jac(G)$ can be found in papers [1], [2], [3], [4], [8], [13], [16].

We denote the vertex and edge set of G by $V(G)$ and $E(G)$, respectively. Given $u, v \in V(G)$, we set a_{uv} to be equal to the number of edges between vertices u and v . The matrix $A = A(G) = \{a_{uv}\}_{u,v \in V(G)}$, called the *adjacency matrix* of the graph G . The degree $d(v)$ of a vertex $v \in V(G)$ is defined by $d(v) = \sum_u a_{uv}$. Let $D = D(G)$ be the diagonal matrix indexed by the elements of $V(G)$ with $d_{vv} = d(v)$. Matrix $L = L(G) = D(G) - A(G)$ is called the *Laplacian matrix*, or simply *Laplacian*, of the graph G .

Recall [8] the following useful relation between the structure of the Laplacian matrix and the Jacobian of a graph G . Consider the Laplacian $L(G)$ as a homomorphism $\mathbb{Z}^{|V|} \rightarrow \mathbb{Z}^{|V|}$, where $|V| = |V(G)|$ is the number of vertices in G . The cokernel $\text{coker}(L(G)) = \mathbb{Z}^{|V|}/\text{im}(L(G))$ — is an abelian group. Let

$$\text{coker}(L(G)) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \dots \oplus \mathbb{Z}_{d_{|V|}}$$

be its Smith normal form satisfying the conditions $d_i | d_{i+1}$, ($1 \leq i \leq |V|$). If the graph is connected, then the groups $\mathbb{Z}_{d_1}, \mathbb{Z}_{d_2}, \dots, \mathbb{Z}_{d_{|V|-1}}$ — are finite, and $\mathbb{Z}_{d_{|V|}} = \mathbb{Z}$. In this case,

$$Jac(G) \cong \mathbb{Z}_{t_1} \oplus \mathbb{Z}_{t_2} \oplus \dots \oplus \mathbb{Z}_{d_{|V|-1}}$$

is the Jacobian of the graph G . In other words, $Jac(G)$ is isomorphic to the torsion subgroup of the cokernel $\text{coker}(L(G))$.

Let M be an integer $n \times n$ matrix, then we can interpret M as a homomorphism from \mathbb{Z}^n to \mathbb{Z}^n . In this interpretation M has a kernel $\ker M$, an image $\text{im } M$, and a cokernel $\text{coker } M = \mathbb{Z}^n / \text{im } M$. We emphasize that $\text{coker } M$ of the matrix M coincides with its Smith normal form.

In what follows, by I_n we denote the identity matrix of order n .

We call an $n \times n$ matrix *circulant*, and denote it by $\text{circ}(a_0, a_1, \dots, a_{n-1})$ if it is of the form

$$\text{circ}(a_0, a_1, \dots, a_{n-1}) = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{pmatrix}.$$

Recall [21] that the eigenvalues of matrix $C = \text{circ}(a_0, a_1, \dots, a_{n-1})$ are given by the following simple formulas $\lambda_j = p(\varepsilon_n^j)$, $j = 0, 1, \dots, n-1$ where $p(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ and ε_n is the order n primitive root of the unity. Moreover, the circulant matrix $C = p(T)$, where $T = \text{circ}(0, 1, 0, \dots, 0)$ is the matrix representation of the shift operator $T : (x_0, x_1, \dots, x_{n-2}, x_{n-1}) \rightarrow (x_1, x_2, \dots, x_{n-1}, x_0)$.

By ([6], lemma 2.1) the $2n \times 2n$ adjacency matrix of the I -graph $I(n, k, l)$ has the following block form

$$A(I(n, k, l)) = \begin{pmatrix} C_n^k & I_n \\ I_n & C_n^l \end{pmatrix},$$

where C_n^k is the $n \times n$ circulant matrix of the form $C_n^k = \text{circ}(\underbrace{0, \dots, 0}_{k \text{ times}}, 1, 0, \dots, 0, 1, \underbrace{0, \dots, 0}_{k-1 \text{ times}})$.

Denote by $L = L(I(n, k, l))$ the Laplacian of $I(n, k, l)$. Since the graph $I(n, k, l)$ is three-valent, we have

$$L = 3I_{2n} - A(I(n, k, l)) = \begin{pmatrix} 3I_n - C_n^k & -I_n \\ -I_n & 3I_n - C_n^l \end{pmatrix}.$$

3 Cokernels of linear operators

Let $P(z)$ be a bimonic integer Laurent polynomial. That is $P(z) = z^p + a_1z^{p+1} + \dots + a_{s-1}z^{p+s-1} + z^{p+s}$ for some integers $p, a_1, a_2, \dots, a_{s-1}$ and some positive integer s . Introduce the following companion matrix \mathcal{A} for the polynomial $P(z) : \mathcal{A} = \left(\begin{array}{c|c} 0 & I_{s-1} \\ \hline -1, -a_1, \dots, -a_{s-1} \end{array} \right)$, where I_{s-1} is the identity $(s-1) \times (s-1)$ matrix. We will use the following properties of \mathcal{A} . Note that $\det \mathcal{A} = (-1)^s$. Hence \mathcal{A} is invertible and inverse matrix \mathcal{A}^{-1} is also integer matrix. The characteristic polynomial of \mathcal{A} coincides with $z^{-p}P(z)$.

Let $\mathbb{A} = \langle \alpha_j, j \in \mathbb{Z} \rangle$ be a free Abelian group freely generated by elements $\alpha_j, j \in \mathbb{Z}$. Each element of \mathbb{A} is a linear combination $\sum_j c_j \alpha_j$ with integer coefficients c_j .

Define the shift operator $T : \mathbb{A} \rightarrow \mathbb{A}$ as a \mathbb{Z} -linear operator acting on generators of \mathbb{A} by the rule $T : \alpha_j \rightarrow \alpha_{j+1}, j \in \mathbb{Z}$. Then T is an endomorphism of \mathbb{A} . Let $P(z)$ be a arbitrary Laurent polynomial with integer coefficients, then $A = P(T)$ is also an endomorphism of \mathbb{A} . Since A is a linear combination of powers of T , the action of A on generators α_j can be given by the infinite set of linear transformations $A : \alpha_j \rightarrow \sum_i a_{ij} \alpha_i, j \in \mathbb{Z}$. Here all sums under consideration

are finite. We set $\beta_i = \sum_j a_{ij}\alpha_j$. Then $\text{im } A$ is a subgroup of \mathbb{A} generated by $\beta_i, i \in \mathbb{Z}$. Hence, $\text{coker } A = \mathbb{A}/\text{im } A$ is an abstract Abelian group $\langle x_i, i \in \mathbb{Z} \mid \sum_i a_{ij}x_i = 0, j \in \mathbb{Z} \rangle$ generated by $x_i, i \in \mathbb{Z}$ with the set of defining relations $\sum_i a_{ij}x_i = 0, j \in \mathbb{Z}$. Here x_j are images of α_j under the canonical homomorphism $\mathbb{A} \rightarrow \mathbb{A}/\text{im } A$. Since T and $A = P(T)$ commute, subgroup $\text{im } A$ is invariant under the action of T . Hence, the actions of T and A are well defined on the factor group $\mathbb{A}/\text{im } A$ and are given by $T : x_j \rightarrow x_{j+1}$ and $A : x_j \rightarrow \sum_i a_{ij}x_i$ respectively.

This allows to present the group $\mathbb{A}/\text{im } A$ as follows $\langle x_i, i \in \mathbb{Z} \mid P(T)x_j = 0, j \in \mathbb{Z} \rangle$. In a similar way, given a set $P_1(z), P_2(z), \dots, P_s(z)$ of Laurent polynomials with integer coefficients, one can define the group $\langle x_i, i \in \mathbb{Z} \mid P_1(T)x_j = 0, P_2(T)x_j = 0, \dots, P_s(T)x_j = 0, j \in \mathbb{Z} \rangle$.

We will use the following lemma.

Lemma 3.1 *Let $T : \mathbb{A} \rightarrow \mathbb{A}$ be the shift operator. Consider endomorphisms A and B of the group \mathbb{A} given by the formulas $A = P(T), B = Q(T)$, where $P(z)$ and $Q(z)$ are Laurent polynomials with integer coefficients. Then $B : \mathbb{A} \rightarrow \mathbb{A}$ induces an endomorphism $B|_{\text{coker } A}$ of the group $\text{coker } A = \mathbb{A}/\text{im } A$ defined by $B|_{\text{coker } A}(\alpha + \text{Im } A) = B(\alpha) + \text{Im } A, \alpha \in \mathbb{A}$. Furthermore*

$$\langle x_i, i \in \mathbb{Z} \mid A(T)x_j = 0, B(T)x_j = 0, j \in \mathbb{Z} \rangle \cong \text{coker } A / \text{im}(B|_{\text{coker } A}) \cong \text{coker}(B|_{\text{coker } A}).$$

Proof: The images $\text{im } A$ and $\text{im } B$ are subgroups in \mathbb{A} . Denote by $\langle \text{im } A, \text{im } B \rangle$ the subgroup generated by elements of $\text{im } A$ and $\text{im } B$. Since $P(z)$ and $Q(z)$ are Laurent polynomials, the operators $A = P(T)$ and $B = Q(T)$ do commute. Hence, subgroup $\text{im } A$ is invariant under endomorphism B . Indeed for any $y = Ax \in \text{im } A$, we have $By = B(Ax) = A(Bx) \in \text{im } A$. This means that $B : \mathbb{A} \rightarrow \mathbb{A}$ induces an endomorphism of the group $\text{coker } A = \mathbb{A}/\text{im } A$. We denote this endomorphism by $B|_{\text{coker } A}$. We note that the Abelian group $\langle x_i, i \in \mathbb{Z} \mid A(T)x_j = 0, B(T)x_j = 0, j \in \mathbb{Z} \rangle$ is naturally isomorphic to $\mathbb{A}/\langle \text{im } A, \text{im } B \rangle$. So we have

$$\mathbb{A}/\langle \text{im } A, \text{im } B \rangle \cong (\mathbb{A}/\text{im } A) / \text{im}(B|_{\text{coker } A}) \cong \text{coker } A / \text{im}(B|_{\text{coker } A}) \cong \text{coker}(B|_{\text{coker } A}).$$

The lemma is proved. □

4 Jacobian group for the I -graph $I(n, k, l)$

In this section we prove one of the main results of the paper. We start with the following theorem.

Theorem 4.1 *Let $L = L(I(n, k, l))$ be the Laplacian of the I -graph $I(n, k, l)$. Then*

$$\text{coker } L \cong \text{coker}(\mathcal{A}^n - I),$$

where \mathcal{A} is $2(k+l) \times 2(k+l)$ companion matrix for the Laurent polynomial

$$(3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1.$$

Proof. Let L be the Laplacian matrix of the graph $I(n, k, l)$. Then, as it was mentioned above, L is a $2n \times 2n$ matrix of the form

$$L = \begin{pmatrix} 3I_n - C_n^k & -I_n \\ -I_n & 3I_n - C_n^l \end{pmatrix},$$

where $C_n^k = \text{circ}(\underbrace{0, \dots, 0}_{k \text{ times}}, 1, 0, \dots, 0, 1, \underbrace{0, \dots, 0}_{k-1 \text{ times}})$.

Consider L as a \mathbb{Z} -linear operator $L : \mathbb{Z}^{2n} \rightarrow \mathbb{Z}^{2n}$. In this case, $\text{coker}(L)$ is an abstract Abelian group generated by elements $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ satisfying the system of linear equations $3x_j - x_{j-k} - x_{j+k} - y_j = 0, 3y_j - y_{j-l} - y_{j+l} - x_j = 0$ for any $j = 1, \dots, n$. Here the indices are considered modulo n . By the property mentioned in Section 2, the Jacobian of the graph $I(n, k, l)$ is isomorphic to the finite part of cokernel of the operator L .

To study the structure of $\text{coker}(L)$ we extend the list of generators to the two bi-infinite sequences of elements $(x_j)_{j \in \mathbb{Z}} = (\dots, x_{-1}, x_0, x_1, x_2, \dots)$ and $(y_j)_{j \in \mathbb{Z}} = (\dots, y_{-1}, y_0, y_1, y_2, \dots)$ setting $x_{j+mn} = x_j$ and $y_{j+mn} = y_j$ for any $m \in \mathbb{Z}$. Then we have the following representation for cokernel of L :

$$\begin{aligned} \text{coker}(L) = \langle x_i, y_i, i \in \mathbb{Z} | & 3x_j - x_{j+k} - x_{j-k} - y_j = 0, \\ & 3y_j - y_{j+l} - y_{j-l} - x_j = 0, x_{j+n} = x_j, y_{j+n} = y_j, j \in \mathbb{Z} \rangle. \end{aligned}$$

Let T be the shift operator defined by the rule $T : x_j \rightarrow x_{j+1}, y_j \rightarrow y_{j+1}, j \in \mathbb{Z}$. Consider the operator $P(T)$ defined by $P(T) = (3 - T^k - T^{-k})(3 - T^l - T^{-l}) - 1$. We use the operator notation from section 3 to represent the cokernel of L . Then we have

$$\begin{aligned} \text{coker}(L) &= \langle x_i, y_i, i \in \mathbb{Z} | (3 - T^k - T^{-k})x_j = y_j, (3 - T^l - T^{-l})y_j = x_j, T^n x_j = x_j, T^n y_j = y_j, j \in \mathbb{Z} \rangle \\ &= \langle x_i, i \in \mathbb{Z} | (3 - T^l - T^{-l})(3 - T^k - T^{-k})x_j = x_j, T^n x_j = x_j, j \in \mathbb{Z} \rangle \\ &= \langle x_i, i \in \mathbb{Z} | ((3 - T^k - T^{-k})(3 - T^l - T^{-l}) - 1)x_j = 0, (T^n - 1)x_j = 0, j \in \mathbb{Z} \rangle \\ &= \langle x_i, i \in \mathbb{Z} | P(T)x_j = 0, (T^n - 1)x_j = 0, j \in \mathbb{Z} \rangle. \end{aligned}$$

To finish the proof, we apply Lemma 3.1 to the operators $A = P(T)$ and $B = Q(T) = T^n - 1$.

Since the Laurent polynomial $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$ is bimonic it can be represented in the form $P(z) = z^{-k-l} + a_1 z^{-k-l+1} + \dots + a_{2k+2l-1} z^{k+l-1} + z^{k+l}$, where $a_1, a_2, \dots, a_{2k+2l-1}$ are integers. Then the companion matrix \mathcal{A} is $\left(\begin{array}{c|c} 0 & I_{2k+2l-1} \\ \hline -1, -a_1, \dots, -a_{2k+2l-1} \end{array} \right)$.

It is easy to see that $\det \mathcal{A} = 1$ and its inverse \mathcal{A}^{-1} is also integer matrix.

For convenience we set $s = 2k + 2l$ to be the size of matrix \mathcal{A} .

Note that for any $j \in \mathbb{Z}$ the relations $P(T)x_j = 0$ can be rewritten as $x_{j+s} = -x_j - a_1 x_{j+1} - \dots - a_{s-1} x_{j+s-1}$. Let $\mathbf{x}_j = (x_{j+1}, x_{j+2}, \dots, x_{j+s})^t$ be s -tuple of generators $x_{j+1}, x_{j+2}, \dots, x_{j+s}$. Then the relation $P(T)x_j = 0$ is equivalent to $\mathbf{x}_j = \mathcal{A} \mathbf{x}_{j-1}$. Hence, we have $\mathbf{x}_1 = \mathcal{A} \mathbf{x}_0$ and $\mathbf{x}_{-1} = \mathcal{A}^{-1} \mathbf{x}_0$, where $\mathbf{x}_0 = (x_1, x_2, \dots, x_s)^t$. So, $\mathbf{x}_j = \mathcal{A}^j \mathbf{x}_0$ for any $j \in \mathbb{Z}$. Conversely, the latter implies $\mathbf{x}_j = \mathcal{A} \mathbf{x}_{j-1}$ and, as a consequence, $P(T)x_j = 0$ for all $j \in \mathbb{Z}$.

Let $\mathbb{A} = \langle \alpha_j, j \in \mathbb{Z} \rangle$ be the Abelian group freely generated by elements $\alpha_j, j \in \mathbb{Z}$. As in Lemma 3.1, we consider operator $A = P(T)$ as an endomorphism of the group \mathbb{A} . Then $\text{coker } A = \mathbb{A}/\text{im } A$ as an abstract Abelian group has the following representation $\langle \bar{x}_i, i \in \mathbb{Z} | P(T)\bar{x}_j = 0, j \in \mathbb{Z} \rangle$. Here \bar{x}_j are images of generators α_j under the canonical homomorphism $\mathbb{A} \rightarrow \mathbb{A}/\text{im } A$.

Our present aim is to show that $\text{coker } A \cong \mathbb{Z}^s$. Then we describe the action of the endomorphism $B|_{\text{coker } A}$ on the $\text{coker } A$. Now we have the following representation of $\text{coker } A$.

$$\begin{aligned} \text{coker } A &= \langle \bar{x}_i, i \in \mathbb{Z} | P(T)\bar{x}_j = 0, j \in \mathbb{Z} \rangle = \\ &= \langle \bar{x}_j, j \in \mathbb{Z} | \bar{x}_\ell + a_1\bar{x}_{\ell+1} + \dots + a_{s-1}\bar{x}_{\ell+s-1} + \bar{x}_{\ell+s} = 0, \ell \in \mathbb{Z} \rangle \\ &= \langle \bar{x}_j, j \in \mathbb{Z} | (\bar{x}_{\ell+1}, \bar{x}_{\ell+2}, \dots, \bar{x}_{\ell+s})^t = \mathcal{A}(\bar{x}_\ell, \bar{x}_{\ell+1}, \dots, \bar{x}_{\ell+s-1})^t, \ell \in \mathbb{Z} \rangle \\ &= \langle \bar{x}_j, j \in \mathbb{Z} | (\bar{x}_{\ell+1}, \bar{x}_{\ell+2}, \dots, \bar{x}_{\ell+s})^t = \mathcal{A}^\ell(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_s)^t, \ell \in \mathbb{Z} \rangle \\ &= \langle \bar{x}_1, \bar{x}_2, \dots, \bar{x}_s | \emptyset \rangle \cong \mathbb{Z}^s. \end{aligned}$$

Since the operators $A = P(T)$ and T commute, the action $T|_{\text{coker } A} : \bar{x}_j \rightarrow \bar{x}_{j+1}, j \in \mathbb{Z}$ on the $\text{coker } A$ is well defined. Now we describe the action of $T|_{\text{coker } A}$ on the set of generators $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_s$. For any $i = 1, \dots, s-1$, we have $T|_{\text{coker } A}(\bar{x}_i) = \bar{x}_{i+1}$ and $T|_{\text{coker } A}(\bar{x}_s) = \bar{x}_{s+1} = -\bar{x}_1 - a_1\bar{x}_2 - \dots - a_{s-2}\bar{x}_{s-1} - a_{s-1}\bar{x}_s$. Hence, the action of $T|_{\text{coker } A}$ on the $\text{coker } A$ is given by the matrix \mathcal{A} . Considering \mathcal{A} as an endomorphism of the $\text{coker } A$, we can write $T|_{\text{coker } A} = \mathcal{A}$. Finally, $B|_{\text{coker } A} = Q(T|_{\text{coker } A}) = Q(\mathcal{A})$. Applying Lemma 3.1, we finish the proof of the theorem. \square

Corollary 4.2 *The Jacobian group $\text{Jac}(I(n, k, l))$ of the I -graph $I(n, k, l)$ is isomorphic to the torsion subgroup of $\text{coker}(\mathcal{A}^n - I)$, where \mathcal{A} is the companion matrix for the Laurent polynomial $(3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$.*

The Corollary 4.2 gives a simple way to find Jacobian group $\text{Jac}(I(n, k, l))$ for small values of k and sufficiently large numbers n . The numerical results are given in the Tables 2 and 3.

5 Counting the number of spanning trees for the I -graph $I(n, k, l)$

Theorem 5.1 *The number of spanning trees of the I -graph $I(n, k, l)$ is given by the formula*

$$\tau_{k,l}(n) = (-1)^{(n-1)(k+l)} n \prod_{s=1}^{k+l-1} \frac{T_n(w_s) - 1}{w_s - 1},$$

where $w_s, s = 1, 2, \dots, k+l-1$ are roots of the order $k+l-1$ algebraic equation

$$\frac{(3 - 2T_k(w))(3 - 2T_l(w)) - 1}{w - 1} = 0,$$

and $T_j(w)$ is the Chebyshev polynomial of the first kind.

Proof. By the celebrated Kirchhoff theorem, the number of spanning trees $\tau_{k,l}(n)$ is equal to the product of nonzero eigenvalues of the Laplacian of a graph $I(n, k, l)$ divided by the number of its vertices $2n$. To investigate the spectrum of Laplacian matrix we note that matrix $C_n^k = T^k + T^{-k}$, where $T = \text{circ}(0, 1, \dots, 0)$ is the $n \times n$ shift operator. The latter equality easily follows from the identity $T^n = I_n$. Hence,

$$L = \begin{pmatrix} 3I_n - T^k - T^{-k} & -I_n \\ -I_n & 3I_n - T^l - T^{-l} \end{pmatrix}.$$

The eigenvalues of circulant matrix T are ε_n^j , where $\varepsilon_n = e^{\frac{2\pi i}{n}}$. Since all eigenvalues of T are distinct, the matrix T is conjugate to the diagonal matrix $\mathbb{T} = \text{diag}(1, \varepsilon_n, \dots, \varepsilon_n^{n-1})$, where diagonal entries of $\text{diag}(1, \varepsilon_n, \dots, \varepsilon_n^{n-1})$ are $1, \varepsilon_n, \dots, \varepsilon_n^{n-1}$. To find spectrum of L , without loss of generality, one can assume that $T = \mathbb{T}$. Then the $n \times n$ blocks of L are diagonal matrices. This essentially simplifies the problem of finding eigenvalues of L . Indeed, let λ be an eigenvalue of L and $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$ be the corresponding eigenvector. Then we have the following system of equations

$$\begin{cases} (3I_n - T^k - T^{-k})x - y &= \lambda x \\ -x + (3I_n - T^l - T^{-l})y &= \lambda y \end{cases}.$$

From here we conclude that $y = (3I_n - T^k - T^{-k})x - \lambda x = (3 - \lambda - T^k - T^{-k})x$. Substituting y in the second equation, we have $((3 - \lambda - T^l - T^{-l})(3 - \lambda - T^k - T^{-k}) - 1)x = 0$.

Recall the matrices under consideration are diagonal and the $(j+1, j+1)$ -th entry of T is equal to ε_n^j . Therefore, we have $((3 - \lambda - \varepsilon_n^{jk} - \varepsilon_n^{-jk})(3 - \lambda - \varepsilon_n^{jl} - \varepsilon_n^{-jl}) - 1)x_{j+1} = 0$ and $y_{j+1} = (3 - \lambda - \varepsilon_n^{jl} - \varepsilon_n^{-jl})x_{j+1}$.

So, for any $j = 0, \dots, n-1$ the matrix L has two eigenvalues, say $\lambda_{1,j}$ and $\lambda_{2,j}$ satisfying the quadratic equation $(3 - \lambda - \varepsilon_n^{jk} - \varepsilon_n^{-jk})(3 - \lambda - \varepsilon_n^{jl} - \varepsilon_n^{-jl}) - 1 = 0$. The corresponding eigenvectors are (x, y) , where $x = \mathbf{e}_{j+1} = (0, \dots, \underbrace{1}_{(j+1)\text{-th}}, \dots, 0)$ and $y = (3 - \lambda - T^k - T^{-k})\mathbf{e}_{j+1}$. In particular,

if $j = 0$ for $\lambda_{1,0}, \lambda_{2,0}$ we have $(1 - \lambda)(1 - \lambda) - 1 = \lambda(\lambda - 2) = 0$. That is, $\lambda_{1,0} = 0$ and $\lambda_{2,0} = 2$. Since $\lambda_{1,j}$ and $\lambda_{2,j}$ are roots of the same quadratic equation, we obtain $\lambda_{1,j}\lambda_{2,j} = P(\varepsilon_n^j)$, where $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$.

Now we have

$$\tau_{k,l}(n) = \frac{1}{2n} \lambda_{2,0} \prod_{j=1}^{n-1} \lambda_{1,j} \lambda_{2,j} = \frac{1}{n} \prod_{j=1}^{n-1} \lambda_{1,j} \lambda_{2,j} = \frac{1}{n} \prod_{j=1}^{n-1} P(\varepsilon_n^j).$$

To continue we need the following lemma.

Lemma 5.2 *The following identity holds*

$$(3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1 = (3 - 2T_k(w))(3 - 2T_l(w)) - 1,$$

where $T_k(w)$ is the Chebyshev polynomial of the first kind and $w = \frac{1}{2}(z + z^{-1})$. Moreover, if k and l are coprime then all roots of the Laurent polynomial $(3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$ counted with multiplicities are $1, 1, z_1, 1/z_1, \dots, z_{k+l-1}, 1/z_{k+l-1}$, where we have $|z_s| \neq 1$, $s = 1, 2, \dots, k+l-1$. So, the right-hand polynomial has the roots $1, w_1, \dots, w_{k+l-1}$, where $w_s \neq 1$ for all $s = 1, 2, \dots, k+l-1$.

Proof. Let us substitute $z = e^{i\varphi}$. It is easy to see that $w = \frac{1}{2}(z + z^{-1}) = \cos \varphi$, so we have $T_k(w) = \cos(k \arccos w) = \cos(k\varphi)$. Then the first statement of the lemma is equivalent to the following trigonometric identity

$$(3 - 2\cos(k\varphi))(3 - 2\cos(l\varphi)) - 1 = (3 - 2T_k(w))(3 - 2T_l(w)) - 1.$$

To prove the second statement of the lemma we suppose that the Laurent polynomial $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$ has a root z_0 such that $|z_0| = 1$. Then $z_0 = e^{i\varphi_0}$, $\varphi_0 \in \mathbb{R}$. Now we

have $(3 - 2\cos(k\varphi_0))(3 - 2\cos(l\varphi_0)) - 1 = 0$. Since $3 - 2\cos(k\varphi_0) \geq 1$ and $3 - 2\cos(l\varphi_0) \geq 1$, the equation holds if and only if $\cos(k\varphi_0) = 1$ and $\cos(l\varphi_0) = 1$. So $k\varphi_0 = 2\pi s_0$ and $\cos(l\varphi_0) = 2\pi t_0$ for some integer s_0 and t_0 . As k and l are relatively prime, there exist two integers p and q such that $kp + ql = 1$. Hence $\varphi_0 = \varphi_0(kp + ql) = 2\pi(ps_0 + qt_0) \in 2\pi\mathbb{Z}$. As a result, $z_0 = e^{i\varphi_0} = 1$. Now we have to show that the multiplicity of the root $z_0 = 1$ is 2. Indeed, $P(1) = P'(1) = 0$ and $P''(1) = -2(k^2 + l^2) \neq 0$. \square

Let us set $H(z) = \prod_{s=1}^m (z - z_s)(z - z_s^{-1})$, where $m = k + l - 1$ and z_s are roots of $P(z)$ different from 1. Then by Lemma 5.2, we have $P(z) = \frac{(z-1)^2}{z^{k+l}} H(z)$.

Lemma 5.3 *Let $H(z) = \prod_{s=1}^m (z - z_s)(z - z_s^{-1})$ and $H(1) \neq 0$. Then*

$$\prod_{j=1}^{n-1} H(\varepsilon_n^j) = \prod_{s=1}^m \frac{T_n(w_s) - 1}{w_s - 1},$$

where $w_s = \frac{1}{2}(z_s + z_s^{-1})$, $s = 1, \dots, m$ and $T_n(x)$ is the Chebyshev polynomial of the first kind.

Proof. It is easy to check that $\prod_{j=1}^{n-1} (z - \varepsilon_n^j) = \frac{z^n - 1}{z - 1}$ if $z \neq 1$. Also we note that $\frac{1}{2}(z^n + z^{-n}) = T_n(\frac{1}{2}(z + z^{-1}))$. By the substitution $z = e^{i\varphi}$, the latter follows from the evident identity $\cos(n\varphi) = T_n(\cos \varphi)$. Then we have

$$\begin{aligned} \prod_{j=1}^{n-1} H(\varepsilon_n^j) &= \prod_{j=1}^{n-1} \prod_{s=1}^m (\varepsilon_n^j - z_s)(\varepsilon_n^j - z_s^{-1}) \\ &= \prod_{s=1}^m \prod_{j=1}^{n-1} (z_s - \varepsilon_n^j)(z_s^{-1} - \varepsilon_n^j) \\ &= \prod_{s=1}^m \frac{z_s^n - 1}{z_s - 1} \frac{z_s^{-n} - 1}{z_s^{-1} - 1} = \prod_{s=1}^m \frac{T_n(w_s) - 1}{w_s - 1}. \end{aligned}$$

\square

Note that $\prod_{j=1}^{n-1} (1 - \varepsilon_n^j) = \lim_{z \rightarrow 1} \prod_{j=1}^{n-1} (z - \varepsilon_n^j) = \lim_{z \rightarrow 1} \frac{z^n - 1}{z - 1} = n$ and $\prod_{j=1}^{n-1} \varepsilon_n^j = (-1)^{n-1}$. As a result, taking into account Lemma 5.2 and Lemma 5.3, we obtain

$$\begin{aligned} \tau_{k,l}(n) &= \frac{1}{n} \prod_{j=1}^{n-1} P(\varepsilon_n^j) = \frac{1}{n} \prod_{j=1}^{n-1} \frac{(\varepsilon_n^j - 1)^2}{(\varepsilon_n^j)^{k+l}} H(\varepsilon_n^j) = \frac{(-1)^{(n-1)(k+l)} n^2}{n} \prod_{j=1}^{n-1} H(\varepsilon_n^j) \\ &= (-1)^{(n-1)(k+l)} n \prod_{s=1}^{k+l-1} \frac{T_n(w_s) - 1}{w_s - 1}. \end{aligned}$$

\square

Corollary 5.4 $\tau_{k,l}(n) = n \left| \prod_{s=1}^{k+l-1} U_{n-1}(\sqrt{\frac{1+w_s}{2}}) \right|^2$, where $w_s, s = 1, 2, \dots, k$ are the same as in Theorem 5.1 and $U_{n-1}(w)$ is the Chebyshev polynomial of the second kind.

Proof. Follows from the identity $\frac{T_n(w)-1}{w-1} = U_{n-1}^2(\sqrt{\frac{1+w}{2}})$. \square

The following theorem appeared after a fruitful discussion with professor D. Lorenzini.

Theorem 5.5 Let $\tau(n) = \tau_{k,l}(n)$ be the number of spanning trees of the graph $I(n, k, l)$. Then there exist an integer sequence $a(n) = a_{k,l}(n), n \in \mathbb{N}$ such that

- 1° $\tau(n) = n a^2(n)$ when n is odd
- 2° $\tau(n) = 6n a^2(n)$ when n is even and $k + l$ is even,
- 3° $\tau(n) = n a^2(n)$ when n is even and $k + l$ is odd.

Proof. Recall that all nonzero eigenvalues of $I(n, k, l)$ are given by the list $\{\lambda_{2,0}, \lambda_{1,j}, \lambda_{2,j}, j = 1, \dots, n-1\}$. By the Kirchhoff theorem we have $2n\tau(n) = \lambda_{2,0} \prod_{j=1}^{n-1} \lambda_{1,j} \lambda_{2,j}$.

Since $\lambda_{2,0} = 2$, we have $n\tau(n) = \prod_{j=1}^{n-1} \lambda_{1,j} \lambda_{2,j}$. We note that $\lambda_{1,j} \lambda_{2,j} = P(\varepsilon_n^j) = P(\varepsilon_n^{n-j}) = \lambda_{1,n-j} \lambda_{2,n-j}$. So, we get $n\tau(n) = (\prod_{j=1}^{(n-1)/2} \lambda_{1,j} \lambda_{2,j})^2$ if n is odd and $n\tau(n) = \lambda_{1,\frac{n}{2}} \lambda_{2,\frac{n}{2}} (\prod_{j=1}^{n/2-1} \lambda_{1,j} \lambda_{2,j})^2$, if n is even. The value $\lambda_{1,\frac{n}{2}} \lambda_{2,\frac{n}{2}} = P(-1) = (3 - 2(-1)^k)(3 - 2(-1)^l) - 1$ is equal to 24, if k and l are of different parity and 4, if both k and l are odd. The case when both k and l are even is impossible, since k and l are relatively prime.

The graph $I(n, k, l)$ admits a cyclic group of automorphisms isomorphic to \mathbb{Z}_n , which acts freely on the set of spanning trees. Therefore, the value $\tau(n)$ is a multiple of n . So $\frac{\tau(n)}{n}$ is an integer. Hence

- 1° $\frac{\tau(n)}{n} = (\frac{\prod_{j=1}^{(n-1)/2} \lambda_{1,j} \lambda_{2,j}}{n})^2$ when n is odd,
- 2° $\frac{\tau(n)}{n} = (\frac{2 \prod_{j=1}^{n/2-1} \lambda_{1,j} \lambda_{2,j}}{n})^2$ when n is even and $k + l$ is even,
- 3° $\frac{\tau(n)}{n} = 6(\frac{2 \prod_{j=1}^{n/2-1} \lambda_{1,j} \lambda_{2,j}}{n})^2$ when n is even and $k + l$ is odd.

Each algebraic number $\lambda_{i,j}$ comes into both products $\prod_{j=1}^{(n-1)/2} \lambda_{1,j} \lambda_{2,j}$ and $\prod_{j=1}^{n/2-1} \lambda_{1,j} \lambda_{2,j}$ with all its Galois conjugate elements. Therefore, both products are integer numbers. From here we conclude that in equalities 1°, 2° and 3° the value that is squared is a rational number. Because $\frac{\tau(n)}{n}$ is integer and 6 is a squarefree, all these rational numbers are integer. Setting $a(n) = \frac{\prod_{j=1}^{(n-1)/2} \lambda_{1,j} \lambda_{2,j}}{n}$ if n is odd and $a(n) = \frac{2 \prod_{j=1}^{n/2-1} \lambda_{1,j} \lambda_{2,j}}{n}$ if n is even, we finish the proof of the theorem. \square

From now on, we aim to estimate the minimum number of generators for the Jacobian of I -graph $I(n, k, l)$.

Lemma 5.6 For any given I -graph $I(n, k, l)$ the number of spanning trees $\tau(n)$ satisfies the inequality $\tau(n) \geq n^3$.

Proof: Recall that for any $j = 0, \dots, n-1$, the Laplacian matrix L of $I(n, k, l)$ has two eigenvalues, say $\lambda_{1,j}$ and $\lambda_{2,j}$, which are roots of the quadratic equation $Q_j(\lambda) = (3 - \lambda - \varepsilon_n^{jk} - \varepsilon_n^{-jk})(3 - \lambda - \varepsilon_n^{jl} - \varepsilon_n^{-jl}) - 1 = 0$. So, $\lambda_{1,j}\lambda_{2,j} = (3 - \varepsilon_n^{jk} - \varepsilon_n^{-jk})(3 - \varepsilon_n^{jl} - \varepsilon_n^{-jl}) - 1 = P(\varepsilon_n^j)$. Note that $\lambda_{1,0} = 0$ and $\lambda_{2,0} = 2$. Furthermore, $\{\lambda_{1,j}, \lambda_{2,j} \mid j = 0, \dots, n-1\}$ is the set of all eigenvalues of L . The Kirchhoff theorem states the following

$$2n \tau_{k,l}(n) = 2n \tau(n) = \lambda_{2,0} \prod_{j=1}^{n-1} \lambda_{1,j} \lambda_{2,j} = 2 \prod_{j=1}^{n-1} \lambda_{1,j} \lambda_{2,j}.$$

Hence $n\tau(n) = \prod_{j=1}^{n-1} P(\varepsilon_n^j)$, where $P(\varepsilon_n^j) = (3 - 2\cos(\frac{2jk\pi}{n}))(3 - 2\cos(\frac{2jl\pi}{n})) - 1$. It is easy to prove the following trigonometric identity

$$(3 - 2\cos(\frac{2jk\pi}{n}))(3 - 2\cos(\frac{2jl\pi}{n})) - 1 = 4\sin(\frac{jk\pi}{n})^2 + 4\sin(\frac{jl\pi}{n})^2 + 16\sin(\frac{jk\pi}{n})^2 \sin(\frac{jl\pi}{n})^2.$$

Connectedness of I -graph implies $GCD(n, k, l) = 1$. It may happen that $GCD(n, k) = m \neq 1$ and $GCD(n, l) = m' \neq 1$. We will use the notation $n = mq = m'q'$, $k = pm$, $l = p'm'$. We introduce three sets, J , J_k and J_l in the following way $J = \{1, 2, \dots, n-1\}$, $J_k = \{j \mid j = dq, d = 1, \dots, m-1\}$ and $J_l = \{j \mid j = dq', d = 1, \dots, m'-1\}$. If $j \in J_k$ then $\sin(\frac{jk\pi}{n}) = 0$ and if $j \in J_l$ then $\sin(\frac{jl\pi}{n}) = 0$. Now we are going to find a low bound for $\tau(n)$. As $n\tau(n) = \prod_{j=1}^{n-1} P(\varepsilon_n^j)$, we evaluate the product.

$$\begin{aligned} \prod_{j=1}^{n-1} P(\varepsilon_n^j) &= \prod_{j=1}^{n-1} (4\sin(\frac{jk\pi}{n})^2 + 4\sin(\frac{jl\pi}{n})^2 + 16\sin(\frac{jk\pi}{n})^2 \sin(\frac{jl\pi}{n})^2) \\ &\geq \prod_{j \in J_k} 4\sin(\frac{jl\pi}{n})^2 \prod_{j \in J_l} 4\sin(\frac{jk\pi}{n})^2 \prod_{j \in J \setminus (J_k \cup J_l)} 16\sin(\frac{jk\pi}{n})^2 \sin(\frac{jl\pi}{n})^2 \\ &= \prod_{j \in J \setminus J_k} 4\sin(\frac{jk\pi}{n})^2 \prod_{j \in J \setminus J_l} 4\sin(\frac{jl\pi}{n})^2. \end{aligned}$$

Now we analyze individual components of the product. We make use of the following simple identity $\cos(\frac{2jp\pi}{q}) = \cos(\frac{2(j+q)p\pi}{q})$.

$$\begin{aligned} \prod_{j \in J \setminus J_k} 4\sin(\frac{jk\pi}{n})^2 &= \prod_{j \in J \setminus J_k} (2 - 2\cos(\frac{2jk\pi}{n})) = \prod_{j \in J \setminus J_k} (2 - 2\cos(\frac{2jmp\pi}{mq})) \\ &= \prod_{j \in J \setminus J_k} (2 - 2\cos(\frac{2jp\pi}{q})) = \prod_{j=1}^{q-1} (2 - 2\cos(\frac{2jp\pi}{q}))^m. \end{aligned}$$

The Chebyshev polynomial $T_q(x) = \cos(q \arccos(x))$ has the following property. The roots of the equation $T_q(x) - 1 = 0$ are $\cos(\frac{2j\pi}{q})$, $j = 0, 1, \dots, q-1$. Since the leading coefficient of $T_q(x)$ is 2^{q-1} , for $x \neq 1$ we have the identity

$$\prod_{j=1}^{q-1} (2x - 2\cos(\frac{2j\pi}{q})) = \frac{T_q(x) - 1}{x - 1}.$$

As p and q are co-prime, we obtain

$$\prod_{j=1}^{q-1} (2 - 2 \cos(\frac{2jp\pi}{q}))^m = \prod_{j=1}^{q-1} (2 - 2 \cos(\frac{2j\pi}{q}))^m = (\lim_{x \rightarrow 1} \frac{T_q(x) - 1}{x - 1})^m = (q^2)^m = (\frac{n}{m})^{2m}.$$

Hence

$$\prod_{j \in J \setminus J_k} 4 \sin(\frac{jk\pi}{n})^2 = (\frac{n}{m})^{2m}.$$

In a similar way, we obtain

$$\prod_{j \in J \setminus J_l} 4 \sin(\frac{jl\pi}{n})^2 = (\frac{n}{m'})^{2m'}.$$

To get the final result we use the following trivial inequality. For any integers $a \geq 2$ and $b \geq 2$ we have $a^b \geq ab$. Since $q = n/m \geq 2$ and $q' = n/m' \geq 2$, we conclude

$$n\tau(n) = \prod_{j=1}^{n-1} P(\varepsilon_n^j) \geq (\frac{n}{m})^{2m} (\frac{n}{m'})^{2m'} \geq n^2 n^2 = n^4.$$

□

Using Lemma 5.6, one can show the following theorem.

Theorem 5.7 *For any given I -graph $I(n, k, l)$ the minimum number of generators for Jacobian $Jac(I(n, k, l))$ is at least 2 and at most $2k + 2l - 1$. The both bounds are sharp.*

Proof: The upper bound for the number of generators follows from theorem 4.1. Indeed, by this theorem the group $\text{coker}(L(I(n, k, l))) \cong Jac(I(n, k, l)) \oplus \mathbb{Z}$ is generated by $2k + 2l$ elements. One of these generators is needed to generate the infinite cyclic group \mathbb{Z} . Hence $Jac(I(n, k, l))$ is generated by $2k + 2l - 1$ elements.

To get the lower bound we use Lemma 5.6. Let us suppose that $Jac(I(n, k, l))$ is generated by one element. Then it is the cyclic group of order $\tau(n)$. Denote by D be a product of all distinct nonzero eigenvalues of $I(n, k, l)$. By Proposition 2.6 from [8], the order of each element of $Jac(I(n, k, l))$ is divisor of D . Hence, $\tau(n)$ is divisor of D and we have inequality $D \geq \tau(n)$. By the Kirchhoff theorem we have $2n\tau(n) = \lambda_{2,0} \prod_{j=1}^{n-1} \lambda_{1,j} \lambda_{2,j}$. We note that any algebraic number $\lambda_{i,j}$ comes into the product together with its Galois conjugate, so $2n\tau(n)$ is a multiple of D . In particular, $2n\tau(n) \geq D$.

From the proof of Theorem 5.5 we have $n\tau(n) = (\prod_{j=1}^{(n-1)/2} \lambda_{1,j} \lambda_{2,j})^2$, if n is odd and $n\tau(n) = \lambda_{1,\frac{n}{2}} \lambda_{2,\frac{n}{2}} (\prod_{j=1}^{n/2-1} \lambda_{1,j} \lambda_{2,j})^2$, if n is even. Moreover, the value $\lambda_{1,\frac{n}{2}} \lambda_{2,\frac{n}{2}}$ is equal to 24, if k and l are of different parity and 4, if both k and l are odd. The case when both k and l are even is impossible as k and l are relatively prime.

Now, we have $4n\tau(n) = (2 \prod_{j=1}^{(n-1)/2} \lambda_{1,j} \lambda_{2,j})^2$ if n is odd. We note that any algebraic number $\lambda_{i,j}$ comes into the product $\rho = 2 \prod_{j=1}^{(n-1)/2} \lambda_{1,j} \lambda_{2,j}$ together with its Galois conjugate. Therefore,

the product ρ is an integer number and contains all distinct nonzero eigenvalues. Hence ρ is a multiple of D . So, we obtain $4n\tau(n) = \rho^2 \geq D^2 \geq \tau(n)^2$.

Also we get $4n\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}\tau(n) = (2\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j})^2$ if n is even. By a similar argument, taking into account the inequality $24 \geq \lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}$ we obtain $96n\tau(n) \geq 4n\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}\tau(n) \geq D^2 \geq \tau(n)^2$.

As result, by Lemma 5.6 we have $4n \geq \tau(n) \geq n^3$ if n is odd and $96n \geq \tau(n) \geq n^3$ if n is even. For $n \geq 10$ this is impossible. So, the rank of $Jac(I(n, k, l))$ is at least two for all $n \geq 10$. For n less than 10 this statement can be proved by direct calculation. \square

The upper bound $2k+2l-1$ for the number of generators of $Jac(I(n, k, l))$ is attained for graphs $I(17, 2, 3)$ and $I(170, 3, 4)$. See Tables 2 and 3 in section 7.

6 Asymptotic for the number of spanning trees

The asymptotic for the number of spanning trees of the graph $I(n, k, l)$ is given in the following theorem.

Theorem 6.1 *Let $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$. Suppose that k and l are relatively prime and set $A_{k,l} = \prod_{P(z)=0, |z|>1} |z|$. Then the number $\tau_{k,l}(n)$ of spanning trees of the graph $I(n, k, l)$ has the asymptotic*

$$\tau_{k,l}(n) \sim \frac{n}{k^2 + l^2} A_{k,l}^n, \quad n \rightarrow \infty.$$

Proof. By theorem 5.1 we have

$$\tau_{k,l}(n) = (-1)^{(n-1)(k+l)} n \prod_{s=1}^{k+l-1} \frac{T_n(w_s) - 1}{w_s - 1},$$

where $w_s, s = 1, 2, \dots, k+l-1$ are roots of the polynomial $Q(w) = \frac{(3-2T_k(w))(3-2T_l(w))-1}{w-1}$. So we obtain

$$\tau_{k,l}(n) = n \prod_{s=1}^{k+l-1} \left| \frac{T_n(w_s) - 1}{w_s - 1} \right| = n \prod_{s=1}^{k+l-1} |T_n(w_s) - 1| / \prod_{s=1}^{k+l-1} |w_s - 1|.$$

By lemma 5.2 we have $T_n(w_s) = \frac{1}{2}(z_s^n + z_s^{-n})$, where the z_s and $1/z_s$ are roots of the polynomial $P(z)$ with the property $|z_s| \neq 1, s = 1, 2, \dots, k+l-1$. Replacing z_s by $1/z_s$, if it is necessary, we can assume that all $|z_s| > 1$ for all $s = 1, 2, \dots, k+l-1$. Then $T_n(w_s) \sim \frac{1}{2}z_s^n$ as n tends to ∞ . So $|T_n(w_s) - 1| \sim \frac{1}{2}|z_s|^n$ as $n \rightarrow \infty$. Hence

$$\prod_{s=1}^{k+l-1} |T_n(w_s) - 1| \sim \frac{1}{2^{k+l-1}} \prod_{s=1}^{k+l-1} |z_s|^n = \frac{1}{2^{k+l-1}} \prod_{P(z)=0, |z|>1} |z|^n = \frac{1}{2^{k+l-1}} A_{k,l}^n.$$

Now we directly evaluate the quantity $\prod_{s=1}^{k+l-1} |w_s - 1|$. We note that $Q(w) = a_0 w^{k+l-1} + a_1 2^{k+l-2} + \dots + a_{k+l-2} w + a_{k+l-1}$ is an integer polynomial with the leading coefficient $a_0 = 2^{k+l}$. From here we obtain

$$\prod_{s=1}^{k+l-1} |w_s - 1| = \prod_{s=1}^{k+l-1} |1 - w_s| = \left| \frac{1}{a_0} Q(1) \right| = \frac{2(k^2 + l^2)}{2^{k+l}} = \frac{k^2 + l^2}{2^{k+l-1}}.$$

Indeed, $Q(1) = \lim_{w \rightarrow 1} \frac{(3-2T_k(w))(3-2T_l(w))-1}{w-1} = -2T'_k(1)(3-2T_l(1)) - 2T'_l(1)(3-2T_k(1)) = -2kU_{k-1}(1)(3-2T_l(1)) - 2lU_{l-1}(1)(3-2T_k(1)) = -2(k^2 + l^2)$ and $a_0 = 2^{k+l}$.

In order to get the statement of the theorem, we combine the above mentioned results. Then

$$\tau_{k,l}(n) \sim n \frac{A_{k,l}^n}{2^{k+l-1}} / \frac{k^2 + l^2}{2^{k+l-1}} = \frac{n}{k^2 + l^2} A_{k,l}^n \text{ as } n \rightarrow \infty.$$

Remark: It was noted by professor A. Yu. Vesnin that constant $A_{k,l}$ coincides with the Mahler measure of Laurent polynomial $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$. It gives a simple way to calculate $A_{k,l}$ using the following formula

$$A_{k,l} = e^{\int_0^1 \log |P(e^{2\pi i t})| dt}.$$

See, for example, ([31], p. 6) for the proof.

The numerical values for $A_{k,l}$, where k and l are relatively prime numbers $1 \leq k \leq l \leq 9$ will be given in Table 1 in the section 7.

7 Examples and Tables

7.1 Examples

- 1° The Prism graph $I(n, 1, 1)$. We have the following asymptotic $\tau_{1,1}(n) = n(T_n(2) - 1) \sim \frac{n}{2}(2 + \sqrt{3})^n$, $n \rightarrow \infty$.
- 2° The generalized Petersen graph $GP(n, 2) = I(n, 1, 2)$. The the number of spanning trees (see [27]) behaves like $\tau_{1,2}(n) \sim \frac{n}{5} A_{1,2}^n$, $n \rightarrow \infty$, where $A_{1,2} = \frac{7+\sqrt{5}+\sqrt{38+14\sqrt{5}}}{4} \cong 4.39026$.
- 3° The smallest proper I -graph $I(n, 2, 3)$ has the following asymptotic for the number of spanning trees $\tau_{2,3}(n) \sim \frac{n}{13} A_{2,3}^n$, $n \rightarrow \infty$. Here $A_{2,3} \cong 4.84199$ is a suitable root of the algebraic equation $1 - 7x + 13x^2 - 35x^3 + 161x^4 - 287x^5 + 241x^6 - 371x^7 + 577x^8 - 371x^9 + 241x^{10} - 287x^{11} + 161x^{12} - 35x^{13} + 13x^{14} - 7x^{15} + x^{16} = 0$.

Here is the table for asymptotic constants $A_{k,l}$ for relatively prime numbers $1 \leq k \leq l \leq 9$.

Table 1: Asymptotic constants $A_{k,l}$

$k \backslash l$	1	2	3	4	5	6	7	8	9
1	3.7320	4.3902	4.7201	4.8954	4.9953	5.0559	5.0945	5.1203	5.1382
2		-	4.8419	-	5.0249	-	5.1033	-	5.1414
3			-	5.0054	5.0541	-	5.1137	5.1320	-
4				-	5.0802	-	5.1244	-	5.1504
5					-	5.1201	5.1346	5.1461	5.1554
6						-	5.1438	-	-
7							-	5.1589	5.1649
8								-	5.1691

7.2 The tables of Jacobians of I -graphs

Theorem 4.1 is the first step to understand the structure of the Jacobian for $I(n, k, l)$. Also, it gives a simple way for numerical calculations of $\text{Jac}(I(n, k, l))$ for small values of k and l . See Tables 2 and 3 below.

Table 2: Graph $I(n, 2, 3)$

n	$\text{Jac}(I(n, 2, 3))$	$\tau_{2,3}(n) = \text{Jac}(I(n, 2, 3)) $
4	$\mathbb{Z}_7 \oplus \mathbb{Z}_{28}$	196
5	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{95}$	1805
6	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{114}$	2166
7	$\mathbb{Z}_{83} \oplus \mathbb{Z}_{581}$	48223
8	$\mathbb{Z}_{161} \oplus \mathbb{Z}_{1288}$	207368
9	$\mathbb{Z}_{289} \oplus \mathbb{Z}_{2601}$	751689
10	$\mathbb{Z}_{1558} \oplus \mathbb{Z}_{3895}$	6068410
11	$\mathbb{Z}_{1693} \oplus \mathbb{Z}_{18623}$	31528739
12	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{665} \oplus \mathbb{Z}_{7980}$	132667500
13	$\mathbb{Z}_{25} \oplus \mathbb{Z}_{325} \oplus \mathbb{Z}_{325} \oplus \mathbb{Z}_{325}$	858203125
14	$\mathbb{Z}_{17513} \oplus \mathbb{Z}_{245182}$	4293872366
15	$\mathbb{Z}_{37069} \oplus \mathbb{Z}_{556035}$	20611661415
16	$\mathbb{Z}_{84847} \oplus \mathbb{Z}_{1357552}$	115184214544
17	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{23186} \oplus \mathbb{Z}_{394162}$	584898568448
18	$\mathbb{Z}_{400843} \oplus \mathbb{Z}_{7215174}$	2892151991682
19	$\mathbb{Z}_{898243} \oplus \mathbb{Z}_{17066617}$	15329969253931
20	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{19} \oplus \mathbb{Z}_{19} \oplus \mathbb{Z}_{19} \oplus \mathbb{Z}_{5453} \oplus \mathbb{Z}_{109060}$	77502443441780
21	$\mathbb{Z}_{4301807} \oplus \mathbb{Z}_{90337947}$	388616412770229
22	$\mathbb{Z}_{9536669} \oplus \mathbb{Z}_{209806718}$	2000857223542342
23	$\mathbb{Z}_{20949827} \oplus \mathbb{Z}_{481846021}$	10094590780588367
24	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{9192295} \oplus \mathbb{Z}_{220615080}$	50598972420215000
25	$\mathbb{Z}_{101468531} \oplus \mathbb{Z}_{2536713275}$	257396569582449025
26	$\mathbb{Z}_{25} \oplus \mathbb{Z}_{325} \oplus \mathbb{Z}_{8923525} \oplus \mathbb{Z}_{17847050}$	1293976099416406250
27	$\mathbb{Z}_{490309597} \oplus \mathbb{Z}_{13238359119}$	6490894524578165043
28	$\mathbb{Z}_{49} \oplus \mathbb{Z}_{154342069} \oplus \mathbb{Z}_{4321577932}$	32683062689111444092
29	$\mathbb{Z}_{2376466133} \oplus \mathbb{Z}_{68917517857}$	163780147157583236981
30	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{19} \oplus \mathbb{Z}_{275089049} \oplus \mathbb{Z}_{8252671470}$	819549256247415262830
31	$\mathbb{Z}_{11507960491} \oplus \mathbb{Z}_{356746775221}$	4105427794534925793511
32	$\mathbb{Z}_{25318259953} \oplus \mathbb{Z}_{810184318496}$	20512457185525873990688
33	$\mathbb{Z}_{55700389051} \oplus \mathbb{Z}_{1838112838683}$	102383600234281102459833
34	$\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{1915580948} \oplus \mathbb{Z}_{32564876116}$	511022336096582352633856
35	$\mathbb{Z}_{269747901677} \oplus \mathbb{Z}_{9441176558695}$	2546737566070056079431515

Table 3: Graph $I(n, 3, 4)$

n	$\text{Jac}(I(n, 3, 4))$	$\tau_{3,4}(n) = \text{Jac}(I(n, 3, 4)) $
5	$\mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$	2000
6	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{114}$	2166
7	$\mathbb{Z}_{71} \oplus \mathbb{Z}_{497}$	35287
8	$\mathbb{Z}_{73} \oplus \mathbb{Z}_{584}$	42632
9	$\mathbb{Z}_{289} \oplus \mathbb{Z}_{2601}$	751689
10	$\mathbb{Z}_2 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{60} \oplus \mathbb{Z}_{60} \oplus \mathbb{Z}_{60}$	5184000
11	$\mathbb{Z}_{1541} \oplus \mathbb{Z}_{16951}$	26121491
12	$\mathbb{Z}_{11} \oplus \mathbb{Z}_{11} \oplus \mathbb{Z}_{209} \oplus \mathbb{Z}_{2508}$	63424812
13	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{1555} \oplus \mathbb{Z}_{20215}$	785858125
14	$\mathbb{Z}_{16969} \oplus \mathbb{Z}_{237566}$	4031257454
15	$\mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{17410} \oplus \mathbb{Z}_{52230}$	18186486000
16	$\mathbb{Z}_{71321} \oplus \mathbb{Z}_{1141136}$	81386960656
17	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{23186} \oplus \mathbb{Z}_{394162}$	584898568448
18	$\mathbb{Z}_{400843} \oplus \mathbb{Z}_{7215174}$	2892151991682
19	$\mathbb{Z}_{37} \oplus \mathbb{Z}_{37} \oplus \mathbb{Z}_{23939} \oplus \mathbb{Z}_{454841}$	14906272578931
20	$\mathbb{Z}_8 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{120} \oplus \mathbb{Z}_{79080} \oplus \mathbb{Z}_{79080}$	72042006528000
21	$\mathbb{Z}_{4487981} \oplus \mathbb{Z}_{94247601}$	422981442583581
22	$\mathbb{Z}_{10002631} \oplus \mathbb{Z}_{220057882}$	2201157792287542
23	$\mathbb{Z}_{22138559} \oplus \mathbb{Z}_{509186857}$	11272663275719063
24	$\mathbb{Z}_{187} \oplus \mathbb{Z}_{187} \oplus \mathbb{Z}_{259369} \oplus \mathbb{Z}_{6224856}$	56458663080288216
25	$\mathbb{Z}_{2114} \oplus \mathbb{Z}_{52850} \oplus \mathbb{Z}_{52850} \oplus \mathbb{Z}_{52850}$	312061332000250000

The first example of Jacobian $\text{Jac}(I(n, 3, 4))$ with the maximum rank 13:

$$n = 170,$$

$$\text{Jac}(I(170, 3, 4)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4^8 \oplus \mathbb{Z}_{6108} \oplus \mathbb{Z}_{30540} \oplus \mathbb{Z}_{2^2 \cdot 3 \cdot 5 \cdot 103 \cdot 509 \cdot 1699 \cdot 11593 \cdot p \cdot q} \oplus \mathbb{Z}_{2^2 \cdot 3 \cdot 5 \cdot 17 \cdot 103 \cdot 509 \cdot 1699 \cdot 11593 \cdot p \cdot q},$$

and

$$\tau_{3,4}(170) = 2^{25} \cdot 3^4 \cdot 5^3 \cdot 17 \cdot 103^2 \cdot 509^4 \cdot 1699^2 \cdot 11593^2 \cdot p^2 \cdot q^2,$$

where $p = 16901365279286026289$ and $q = 34652587005966540929$.

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